

Boyle's Conjecture and perfect localizations

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Abstract

In this article we study the behaviour of left QI-rings under perfect localizations. We show that a perfect localization of a left QI-ring is a left QI-ring. We prove that Boyle's conjecture is true for left QI-rings with finite Gabriel dimension such that the hereditary torsion theory generated by semisimple modules is perfect. As corollary we get that Boyle's conjecture is true for left QI-rings which satisfy the restricted left socle condition, this result was proved first by C. Faith in [6].

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1 Introduction

Through all this paper R will denote an associative ring with unit element. We will work with unitary left R -modules and the category of modules will be denoted by $R\text{-Mod}$. For general background on module and ring theory we refer the reader to [1], [10], [11] and [12].

Remember that an R -module M is quasi-injective if every morphism $f : N \rightarrow M$ with $N \leq M$ can be extended to an endomorphism of M . Equivalently M is quasi-injective if and only if M is fully invariant in its injective hull. A ring R is called left QI-ring if every quasi-injective left R -module is injective, these rings were introduced by A. Boyle in [2]. Also in [2] is shown that a left QI-ring is left noetherian and left V -ring; recall that a left V -ring is defined as a ring where every simple left module is injective.

In [4] the author introduces two examples of non semisimple left QI-rings. These examples are left hereditary rings, that is, every left ideal is projective.

In [2, Theorem 5] A. Boyle characterizes two-sided hereditary, right noetherian, left QI-rings and she conjectured that every left QI-ring is left hereditary. In [6] C. Faith gave an approach to this conjecture. In [6, Corollary 3] is shown that every left QI-ring is a finite product of simple left QI-rings, so it is enough to formulate Boyle's conjecture for simple left QI-rings.

C. Faith, in [6, Theorem 18], answers in affirmative Boyle's conjecture for simple left QI-rings which satisfy the restricted left socle condition (RLS):

If $I \neq R$ is an essential left ideal, then R/I has non zero socle.

The Theorem 18 in [6] was extended by T. Rizvi and D. Van Huynh in [9]. But the conjecture is still open.

This paper is organized in three sections, the first one is this introduction and the second one concerns to present the necessary preliminaries.

Section 3 is where we develop our work and it contains the main results, we prove that if R is a left QI-ring and τ is a perfect torsion theory then the ring of quotients ${}_{\tau}R$ is left QI (Proposition 3.10). Also, we prove that Boyle's conjecture is true for all those left QI-rings with finite Gabriel dimension such that the hereditary torsion theory generated by the semisimple modules is perfect (Theorem 3.12).

2 Preliminaries

One useful tool to characterize rings, is the hereditary torsion theories. Let us recall the definition of hereditary torsion theory:

Definition 2.1. A pair of nonempty classes of modules $\tau = (\mathfrak{T}, \mathfrak{F})$ is a torsion theory if

1. $\text{Hom}_R(T, F) = 0$ for all $T \in \mathfrak{T}$ and $F \in \mathfrak{F}$.
2. If M is such that $\text{Hom}_R(M, F) = 0$ for all $F \in \mathfrak{F}$ then $M \in \mathfrak{T}$.
3. If N is such that $\text{Hom}_R(T, N) = 0$ for all $T \in \mathfrak{T}$ then $N \in \mathfrak{F}$.

It is said τ is hereditary if \mathfrak{T} is closed under submodules. The set of hereditary torsion theories in $R\text{-Mod}$ is denoted $R\text{-tors}$. The class \mathfrak{T} is called the torsion class and \mathfrak{F} the torsion free class.

It can be proved $R\text{-tors}$ is a frame with order the inclusion of torsion classes, the least and greatest elements of $R\text{-tors}$ are denoted by ξ and χ respectively. Given a class of modules \mathcal{C} there exists the least hereditary torsion theory $\xi(\mathcal{C})$ such that \mathcal{C} is contained in the torsion class and there exist the greatest hereditary torsion theory $\chi(\mathcal{C})$ such that \mathcal{C} is contained in the torsion free class. If $\tau = (\mathfrak{T}, \mathfrak{F})$, the modules in \mathfrak{T} are called τ -torsion modules and the modules in \mathfrak{F} are called τ -torsion free. For more details see [7].

There exist a bijective correspondence between $R\text{-tors}$ and $R\text{-Gab}$, where $R\text{-Gab}$ denotes the set of Gabriel topologies in R . see [12, VI, Theorem 5.1].

An R -module N is called τ -cocritical with $\tau \in R\text{-tors}$ if N is τ -torsion free but every proper factor module is τ -torsion. With this modules can be defined the Gabriel filtration in $R\text{-tors}$ as:

$$\tau_0 = \xi$$

If i is a non limit ordinal:

$$\tau_i = \tau_{i-1} \vee \bigvee \{\xi(N) \mid N \text{ is } \tau_{i-1}\text{-cocritical}\}$$

If i is a limit ordinal:

$$\tau_i = \bigvee_{j < i} \tau_j$$

Since $R\text{-tors}$ is a set it must exist the least ordinal α such that $\tau_\alpha = \tau_{\alpha+\beta}$ for all ordinals β . If $\tau_\alpha = \chi$ then we say R has Gabriel dimension equal to α and we denote it as $Gdim(R) = \alpha$.

The concept of QI-ring can be generalized to modules, one paper in this sense is [5]. In that paper it is shown ([5, Theorem 3.9]) that (in a more general context) a ring R is left QI if and only if R has Gabriel dimension and every hereditary pretorsion class is a hereditary torsion class. A hereditary pretorsion class in $R\text{-Mod}$ is a class of modules closed under submodules, direct sums and quotients.

Let us recall the concept of perfect localization.

Definition 2.2. If $\varphi : R \rightarrow S$ is an epimorphism in the category of rings which makes S into a flat right R -module, then we will call S a left perfect localization of R .

Remark 2.3. Given a hereditary torsion theory $\tau \in R\text{-tors}$ we will say that τ is perfect if the ring of quotients ${}_\tau R$ is a perfect left localization of R .

Remark 2.4. Given $\tau \in R\text{-tors}$, let us denote the localization functor as $\mathbf{q}_\tau : R\text{-Mod} \rightarrow {}_\tau R\text{-Mod}$. If τ is perfect then \mathbf{q}_τ is exact. [12, XI, Proposition 3.4]. We will write ${}_\tau N = \mathbf{q}_\tau(N)$.

3 Left QI-rings and perfect torsion theories

Remark 3.1. Let $\tau \leq \sigma \in R\text{-tors}$. Note that if N is σ -torsion free then $\mathbf{q}_\tau(N)$ is σ -torsion free. In fact, since $\tau \leq \sigma$ then N is τ -torsion free, so we have an essential monomorphism $\psi_N : N \rightarrow \mathbf{q}_\tau(N)$.

Lemma 3.2. Let $\tau \leq \sigma \neq \chi \in R\text{-tors}$ perfect torsion theories with R σ -torsion free. Let $\mathbf{q}_\tau : R\text{-Mod} \rightarrow {}_\tau R\text{-Mod}$ denote the localization functor.

1. If $\sigma = (\mathfrak{T}, \mathfrak{F})$ then $\hat{\sigma} := (\mathbf{q}_\tau(\mathfrak{T}), \mathbf{q}_\tau(\mathfrak{F})) \in {}_\tau R\text{-tors}$.
2. If \mathcal{F}_σ and $\mathcal{F}_{\hat{\sigma}}$ denote the Gabriel topologies in R and ${}_\tau R$ associated to σ and $\hat{\sigma}$ respectively, then:

$$J \in \mathcal{F}_{\hat{\sigma}} \Leftrightarrow J = {}_\tau I \text{ with } I \in \mathcal{F}_\sigma$$

3. ${}_\sigma R$ is $\hat{\sigma}$ -closed (as ${}_\tau R$ -module).

4. There is a ring isomorphism ${}_{\sigma}R \cong {}_{\hat{\sigma}}({}_{\tau}R)$

5. $\hat{\sigma} \in {}_{\tau}R\text{-tors}$ is perfect.

Proof. 1. Let $\tau \leq \sigma \in R\text{-tors}$, with $\sigma = (\mathfrak{T}, \mathfrak{F})$. Then $\hat{\sigma} = (\mathfrak{q}_{\tau}(\mathfrak{T}), \mathfrak{q}_{\tau}(\mathfrak{F}))$ where

$$\mathfrak{q}_{\tau}(\mathfrak{T}) = \{\mathfrak{q}_{\tau}(M) | M \in \mathfrak{T}\}$$

$$\mathfrak{q}_{\tau}(\mathfrak{F}) = \{\mathfrak{q}_{\tau}(N) | N \in \mathfrak{F}\}$$

Let $\mathfrak{q}_{\tau}(M) \in \mathfrak{q}_{\tau}(\mathfrak{T})$ and $\mathfrak{q}_{\tau}(N) \in \mathfrak{q}_{\tau}(\mathfrak{F})$. Since τ is perfect, $\mathfrak{q}_{\tau}(M) = {}_{\tau}R \otimes_R M$. Then

$$\begin{aligned} \text{Hom}_{\tau R}(\mathfrak{q}_{\tau}(M), {}_{\tau}N) &= \text{Hom}_{\tau R}({}_{\tau}R \otimes_R M, {}_{\tau}N) \cong \text{Hom}_R(M, \text{Hom}_{\tau R}({}_{\tau}R, {}_{\tau}N)) \\ &\cong \text{Hom}_R(M, {}_{\tau}N) = 0 \end{aligned}$$

by remark 3.1.

Now, let $\mathfrak{q}_{\tau}(N) = {}_{\tau}N$ an ${}_{\tau}R$ -module such that $\text{Hom}_{\tau R}({}_{\tau}R \otimes_R M, {}_{\tau}N) = 0$ for all $M \in \mathfrak{T}$. Following the above isomorphisms, we get that $\text{Hom}_R(M, {}_{\tau}(N)) = 0$, i.e., ${}_{\tau}N \in \mathfrak{F}$. Since $\tau \leq \sigma$, we have a monomorphism $\psi_N : N \rightarrow {}_{\tau}N$ so $N \in \mathfrak{F}$. Thus ${}_{\tau}N \in \mathfrak{q}_{\tau}(\mathfrak{F})$.

On the other hand, suppose ${}_{\tau}M$ is an ${}_{\tau}R$ -module such that $\text{Hom}_{\tau R}({}_{\tau}M, {}_{\tau}N) = 0$ for all ${}_{\tau}N \in \mathfrak{F}$. Then, we have that $\text{Hom}_R(M, {}_{\tau}N) = 0$. Let $N \in \mathfrak{F}$ and suppose $\text{Hom}_R(M, N) \neq 0$. Hence there exists $0 \neq f : M \rightarrow N$, this implies that $0 \neq \psi_N f : M \rightarrow {}_{\tau}N$. Contradiction. Thus $\text{Hom}_R(M, N) = 0$ for all $N \in \mathfrak{F}$, hence $M \in \mathfrak{T}$.

Thus, we have that $\hat{\sigma}$ is a torsion theory. Let us see it is hereditary.

Let ${}_{\tau}M \in \mathfrak{q}_{\tau}(\mathfrak{T})$ and $K \leq {}_{\tau}M$ an ${}_{\tau}R$ -submodule. There is a monomorphism $\psi : M/\tau(M) \rightarrow {}_{\tau}M$. Consider $K \cap (M/\tau(M))$ which is a σ -torsion R -module. By [12, XI, Proposition 3.7] $\mathfrak{q}_{\tau}(K \cap (M/\tau(M))) = K$. Thus $K \in \mathfrak{q}_{\tau}(\mathfrak{T})$.

$2 \Leftarrow$. Let $I \in \mathcal{F}_{\sigma}$. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & {}_{\tau}I & \longrightarrow & {}_{\tau}R & \longrightarrow & {}_{\tau}(R/I) \longrightarrow 0 \end{array}$$

Since R/I is σ -torsion then ${}_{\tau}(R/I) \in \mathfrak{q}_{\tau}(\mathfrak{T})$ and ${}_{\tau}R/{}_{\tau}I \cong {}_{\tau}(R/I)$. Thus ${}_{\tau}I \in \mathcal{F}_{\hat{\sigma}}$.

\Rightarrow . Let E be an injective R -module such that $\chi(E) = \sigma$. Since E is σ -torsion free then it is τ -torsion free. Hence E is τ -closed, so $E = {}_\tau E$. Let $J \in \mathcal{F}_{\hat{\sigma}}$, since R is τ -torsion free then $J = {}_\tau(J \cap R)$. Therefore:

$$\begin{aligned} \text{Hom}_R(R/J \cap R, E) &= \text{Hom}_R(R/J \cap R, {}_\tau E) \cong \text{Hom}_R(R/J \cap R, \text{Hom}_{{}_\tau R}({}_\tau R, E)) \\ &\cong \text{Hom}_{{}_\tau R}({}_\tau R \otimes_R R/J \cap R, E) \cong \text{Hom}_{{}_\tau R}({}_\tau(R/J \cap R), E) = \text{Hom}_{{}_\tau R}({}_\tau R/J, E) \end{aligned}$$

Since E is σ -torsion free then E is $\hat{\sigma}$ -torsion free but ${}_\tau R/J$ is $\hat{\sigma}$ -torsion. Thus $\text{Hom}_{{}_\tau R}({}_\tau R/J, E) = 0$. This implies $\text{Hom}_R(R/J \cap R, E) = 0$ and hence $J \cap R \in \mathcal{F}_\sigma$.

3. Let us see first that ${}_\sigma R$ is τ -closed (as R -module). By [12, IX, Proposition 1.8] ${}_\sigma R$ is σ -closed, that is, ${}_\sigma R$ is σ -torsion free and \mathcal{F}_σ -injective. Thus ${}_\sigma R$ is τ -torsion free and since $\mathcal{F}_\tau \subseteq \mathcal{F}_\sigma$ then ${}_\sigma R$ is \mathcal{F}_τ -injective.

Since R is σ -torsion free then ${}_\tau R$ is $\hat{\sigma}$ -torsion free and we have following inclusions

$$R \hookrightarrow {}_\tau R \hookrightarrow {}_\sigma R$$

Also, we have to note that this inclusions are essential, so ${}_\sigma R$ is $\hat{\sigma}$ -torsion free. Now, let $J \in \mathcal{F}_{\hat{\sigma}}$ and $g : J \rightarrow {}_\sigma R$ an ${}_\tau R$ -morphism. Then $J = {}_\tau(J \cap R)$ and by (2) $R/J \cap R$ is σ -torsion. If $\psi : R/J \cap R \rightarrow {}_\tau(R/J \cap R) = {}_\tau R/J$ is the canonical homomorphism then $\text{Ker}(\psi)$ and $\text{Coker}(\psi)$ are τ -torsion, therefore they are σ -torsion. So we have the following short exact sequence:

$$0 \rightarrow \frac{R/J \cap R}{\text{Ker}(\psi)} \rightarrow {}_\tau R/J \rightarrow \text{Coker}(\psi) \rightarrow 0$$

Thus, ${}_\tau R/J$ is σ -torsion.

Since ${}_\sigma R$ is \mathcal{F}_σ -injective there exists an R -morphism $\bar{g} : {}_\tau R \rightarrow {}_\sigma R$ such that $\bar{g}|_J = g$. We have that ${}_\sigma R$ is τ -closed, so \bar{g} is an R -morphism between τ -closed R -module, hence \bar{g} is an ${}_\tau R$ -morphism. Thus ${}_\sigma R$ is $\mathcal{F}_{\hat{\sigma}}$ -injective.

4. Since ${}_\sigma R$ is an ${}_\tau R$ -module which is $\hat{\sigma}$ -closed and ${}_\tau R \leq_e {}_\sigma R$ then ${}_\sigma R \cong_{\hat{\sigma}} ({}_\tau R)$.

5. We have ${}_\sigma R \cong_{\hat{\sigma}} ({}_\tau R)$. Let N be an ${}_\sigma R$ -module. Since σ is perfect then $N = {}_\sigma N$ with ${}_R N$ σ -torsion free. Then ${}_\tau N \in \mathbf{q}_\tau(\mathfrak{F})$. Thus ${}_\sigma N$ is $\hat{\sigma}$ -torsion free. By [12, XI, Ex. 6] $\hat{\sigma} \in {}_\tau R\text{-tors}$ is perfect.

□

Lemma 3.3. *Let $\tau \leq \sigma$ be perfect torsion theories in $R\text{-Mod}$ and let $\mathbf{q}_\tau : R\text{-Mod} \rightarrow {}_\tau R\text{-Mod}$ the localization functor.*

1. If M is σ -cocritical then ${}_{\tau}M$ is $\mathfrak{q}_{\tau}(\sigma)$ -cocritical.
2. If an ${}_{\tau}R$ -module K is $\mathfrak{q}_{\tau}(\sigma)$ -cocritical then K as R -module is σ -cocritical.

Proof. 1. Since M is σ -torsion free then ${}_{\tau}M$ is $\mathfrak{q}_{\tau}(\sigma)$ -torsion free. Let $N \leq {}_{\tau}M$ be an ${}_{\tau}R$ -submodule. Since M is σ -torsion free then it is τ -torsion free, so the canonical morphism $\psi_M : M \rightarrow {}_{\tau}M$ is a monomorphism. By [12, IX, Proposition 4.3] $N = {}_{\tau}(N \cap M)$. Since τ is perfect, then \mathfrak{q}_{τ} is exact, hence $\frac{{}_{\tau}M}{N} \cong {}_{\tau}(\frac{M}{N \cap M})$. By hypothesis $\frac{M}{M \cap N}$ is σ -torsion thus $\frac{{}_{\tau}M}{N}$ is $\mathfrak{q}_{\tau}(\sigma)$ -torsion.

2. Let K be a $\mathfrak{q}_{\tau}(\sigma)$ -cocritical. Since K is $\mathfrak{q}_{\tau}(\sigma)$ -torsion free then $K = {}_{\tau}M$ for some M σ -torsion free. So, as R -module K is σ -torsion free. Now, let $L < K$ be an R -submodule such that K/L is σ -torsion free. Since \mathfrak{q}_{τ} is exact $\frac{K}{L} \cong {}_{\tau}(\frac{K}{L})$ but ${}_{\tau}(\frac{K}{L})$ is $\mathfrak{q}_{\tau}(\sigma)$ -torsion free and $\frac{K}{L}$ is $\mathfrak{q}_{\tau}(\sigma)$ -torsion, this is a contradiction. This implies that $N/L = t_{\sigma}(K/L) \neq 0$ and $\frac{K}{N} \cong \frac{K/L}{N/L}$ is σ -torsion free, hence $N = K$. Thus K/L is σ -torsion. \square

Lemma 3.4. *Let R be a ring with finite Gabriel dimension, $Gdim(R) = n$. Let $\{\tau_i\}_{i=0}^n$ be the Gabriel filtration in R -tors with every τ_i perfect. If $\mathfrak{q}_{\tau_1} : R - Mod \rightarrow {}_{\tau_1}R - Mod$ is the localization functor and $\{\omega_j\}$ is the Gabriel filtration in ${}_{\tau_1}R$ -tors, then $\mathfrak{q}_{\tau_1}(\tau_{i+1}) = \omega_i$ for all $0 \leq i$.*

Proof. By induction over i . If $i = 0$ then $\tau_1 = \xi \in {}_{\tau_1}R$ -tors and $\omega_0 = \xi$. Now suppose the result is valid for each natural less than i .

By hypothesis of induction $\mathfrak{q}_{\tau_1}(\tau_i) = \omega_{i-1}$, so

$$\begin{aligned} \omega_i &= \omega_{i-1} \vee \bigvee \{\xi(K) \mid K \text{ is } \omega_{i-1}\text{-cocritical}\} \\ &= \mathfrak{q}_{\tau_1}(\tau_i) \vee \bigvee \{\xi(K) \mid K \text{ is } \mathfrak{q}_{\tau_1}(\tau_i)\text{-cocritical}\} \end{aligned}$$

Let K be a $\mathfrak{q}_{\tau_1}(\tau_i)$ -cocritical, then by Lemma 3.3.2 K as R -module is τ_i -cocritical, hence K is τ_{i+1} -torsion. Thus K is $\mathfrak{q}_{\tau_1}(\tau_{i+1})$ -torsion. Therefore $\omega_i \leq \mathfrak{q}_{\tau_1}(\tau_{i+1})$. By Lemma 3.3.1 if N is τ_i -cocritical then ${}_{\tau_1}N$ is $\mathfrak{q}_{\tau_1}(\tau_i) = \omega_{i-1}$ -cocritical, so ${}_{\tau_1}N$ is ω_i -torsion.

Let L be a ω_i -torsion free. Then L is $\omega_{i-1} = \mathfrak{q}_{\tau_1}(\tau_i)$ -torsion free and hence ${}_RL$ is τ_i -torsion free. If L is not τ_{i+1} -torsion free there exists an R -morphism $0 \neq f : N \rightarrow L$ with N τ_i -cocritical. Then we have a non zero ${}_{\tau_1}R$ -morphism ${}_{\tau_1}f : {}_{\tau_1}N \rightarrow L$ with ${}_{\tau_1}N$ ω_{i-1} -cocritical. Contradiction. Thus L is τ_{i+1} -torsion free. This implies that L is $\mathfrak{q}_{\tau_1}(\tau_{i+1})$ -torsion free. Thus $\mathfrak{q}_{\tau_1}(\tau_{i+1}) \leq \omega_i$. \square

Corollary 3.5. *Let R be a ring with finite Gabriel dimension, $Gdim(R) = n$. Let $\{\tau_i\}_{i=0}^n$ be the Gabriel filtration in R -tors. Suppose that every τ_i is perfect, then $Gdim({}_{\tau_1}R) < n$.*

Proof. If $\{\omega_j\}$ is the Gabriel filtration in ${}_{\tau_1}R$ then, by Lemma 3.4 $\mathfrak{q}_{\tau_1}(\tau_{i+1}) = \omega_i$ for all $0 \leq i$. Since $Gdim(R) = n$ then $\tau_n = \chi$ so $\omega_{n-1} = \mathfrak{q}_{\tau_1}(\tau_n) = \mathfrak{q}_{\tau_1}(\chi) = \chi \in {}_{\tau_1}R$ -tors. This implies that $Gdim({}_{\tau_1}R) \leq n - 1$. \square

Lemma 3.6. *Let $\tau \in R$ -tors and $\{\sigma_i\}_{i \in I} \subseteq R$ -tors be a family of perfect torsion theories such that $\tau \leq \sigma_i$ for all $i \in I$. If $\mathfrak{q}_\tau : R\text{-Mod} \rightarrow {}_\tau R\text{-Mod}$ is the localization functor then*

$$\mathfrak{q}_\tau\left(\bigvee_{i \in I} \sigma_i\right) = \bigvee_{i \in I} \mathfrak{q}_\tau(\sigma_i)$$

Proof. Write $\bigvee_{i \in I} \sigma_i = (\mathfrak{T}_{\bigvee \sigma_i}, \mathfrak{F}_{\bigvee \sigma_i})$. Then

$$\mathfrak{q}_\tau\left(\bigvee_{i \in I} \sigma_i\right) = (\mathfrak{q}_\tau(\mathfrak{T}_{\bigvee \sigma_i}), \mathfrak{q}_\tau(\mathfrak{F}_{\bigvee \sigma_i}))$$

The torsion free class of $\bigvee_{i \in I} \sigma_i$ is described as $\mathfrak{F}_{\bigvee \sigma_i} = \bigcap_{i \in I} \mathfrak{F}_{\sigma_i}$. So, if $\mathfrak{q}_\tau(N) \in \mathfrak{q}_\tau(\mathfrak{F}_{\bigvee \sigma_i})$ then $N \in \bigcap_{i \in I} \mathfrak{F}_{\sigma_i}$, hence $\mathfrak{q}_\tau(N) \in \bigcap_{i \in I} \mathfrak{q}_\tau(\mathfrak{F}_{\sigma_i})$. Thus

$$\bigvee_{i \in I} \mathfrak{q}_\tau(\sigma_i) \leq \mathfrak{q}_\tau\left(\bigvee_{i \in I} \sigma_i\right)$$

Now, suppose $\bigvee_{i \in I} \mathfrak{q}_\tau(\sigma_i) < \mathfrak{q}_\tau\left(\bigvee_{i \in I} \sigma_i\right)$, that is, $\mathfrak{T}_{\bigvee \mathfrak{q}_\tau(\sigma_i)} < \mathfrak{q}_\tau(\mathfrak{T}_{\bigvee \sigma_i})$ then there exists $0 \neq \mathfrak{q}_\tau(N) \in \mathfrak{q}_\tau(\mathfrak{T}_{\bigvee \sigma_i})$ such that $\mathfrak{q}_\tau(N) \in \mathfrak{F}_{\bigvee \mathfrak{q}_\tau(\sigma_i)} = \bigcap_{i \in I} \mathfrak{F}_{\mathfrak{q}_\tau(\sigma_i)}$.

Since $\tau(N/\tau(N)) = 0$ then $\mathfrak{q}_\tau(N) = \mathfrak{q}_\tau(N/\tau(N))$, we have that $N \in \mathfrak{T}_{\bigvee \sigma_i}$ then $N/\tau(N) \in \mathfrak{T}_{\bigvee \sigma_i}$. Therefore, we can assume N is τ -torsion free. By the choice of N there exists $j \in I$ such that $N \notin \mathfrak{F}_{\sigma_j}$, so $\sigma_j(N) = N' \neq 0$. On the other hand, $\mathfrak{q}_\tau(N) \in \bigcap_{i \in I} \mathfrak{F}_{\mathfrak{q}_\tau(\sigma_i)}$ then there exists $N_j \in \mathfrak{F}_{\sigma_j}$ such that $\mathfrak{q}_\tau(N) = \mathfrak{q}_\tau(N_j)$. Since $\tau < \sigma_j$, N_j is τ -torsion free, thus $N_j \leq_e \mathfrak{N}_j = \mathfrak{q}_\tau(N)$. This implies that $0 \neq N' \cap N_j$ but N' is σ_j -torsion and N_j is σ_j -torsion free, this is a contradiction. Thus

$$\bigvee_{i \in I} \mathfrak{q}_\tau(\sigma_i) = \mathfrak{q}_\tau\left(\bigvee_{i \in I} \sigma_i\right)$$

\square

Remark 3.7. In general Lemma 3.4 is not true for infinite ordinals. Let R be a ring with Gabriel dimension, $Gdim(R) = \alpha$, $\omega < \alpha$. Let $\{\tau_i\}_{i=0}^\alpha$ the Gabriel filtration in R -tors and suppose that every τ_i is perfect. Then, by the proof of Lemma 3.4, if $\{w_j\}$ is the Gabriel filtration in ${}_{\tau_1}R$ -tors we have that $\mathfrak{q}_{\tau_1}(\tau_{i+1}) = w_i$ for all $i \in \mathbb{N}$. If ω is the first infinite ordinal, by Lemma 3.6

$$\mathfrak{q}_{\tau_1}(\tau_\omega) = \mathfrak{q}_{\tau_1}\left(\bigvee_{i \in \mathbb{N}} \tau_i\right) = \bigvee_{i \in \mathbb{N}} \mathfrak{q}_{\tau_1}(\tau_i) = \bigvee_{i \in \mathbb{N}} w_{i-1} = w_\omega$$

Definition 3.8. An injective left R -module E is called completely injective if every factor module of E is injective.

The following result is well known, see [1, 18, Ex. 10]

Proposition 3.9. *Let R be a ring. R is left hereditary if and only if every injective module is completely injective.*

Proposition 3.10. *Let R be a left QI-ring and $\tau \in R$ -tors a perfect torsion theory. Then the ring of quotients ${}_\tau R$ is a left QI-ring.*

Proof. Let ${}_\tau A$ be a quasi-injective ${}_\tau R$ -module. Then we can consider A as a τ -torsion free R -module and by [12, IX, Proposition 2.5] $E(A)$ is an injective envelope of ${}_\tau A$ in ${}_\tau R$ -Mod. Hence ${}_\tau A \leq E(A)$ is a fully invariant ${}_\tau R$ -submodule.

Let $f \in \text{End}_R(E(A))$, since $E(A)$ is τ -closed then f is an ${}_\tau R$ -morphism. Then $f({}_\tau A) \leq {}_\tau A$, i.e., ${}_\tau A$ is a quasi-injective R -module. Since R is left QI, ${}_\tau A$ is an injective R -module. Thus by [12, IX, Proposition 2.7] ${}_\tau A$ is an injective ${}_\tau R$ -module. \square

Remark 3.11. Let R be a left QI-ring. Consider the class of all semisimple left R -modules, it is known that this class is a hereditary pretorsion class but since R is left QI then, semisimple modules form a hereditary torsion class [5, Theorem 3.9]. Let us denote the hereditary torsion theory associated to the semisimple torsion class by τ_{ss} . The radical associated to τ_{ss} is Soc .

In the same way, if we consider the pretorsion class of all singular modules it is a hereditary torsion class. We will denote the hereditary torsion class by τ_g and the radical associated by \mathcal{Z} . Notice that if R is a simple ring then τ_g is the unique coatom in R -tors.

Theorem 3.12. *Let R be a (simple) left QI-ring. Suppose that $Gdim(R) = n$ and let $\{\tau_i\}_{i=1}^n$ be the Gabriel filtration in R -tors. Suppose τ_1 is perfect. If ${}_{\tau_1}R$ is left hereditary then R is left hereditary.*

Proof. Since R is a left QI-ring then $\tau_1 = \tau_{ss}$ and by hypothesis $\tau_1 R$ is left hereditary.

Now, let E be an indecomposable non singular injective left R -module. Let $E \rightarrow F$ an epimorphism. Since R is a left noetherian and left V-ring $F = Soc(F) \oplus F'$ where F' is τ_1 -torsion free and $Soc(F)$ is injective. So, to prove R is left hereditary is enough to prove that every factor module F of E with $Soc(F) = 0$ is injective.

Let $\rho : E \rightarrow F$ be an epimorphism such that $Soc(F) = 0$. This implies that $Ker(\rho) \in Sat_{\tau_1}(E)$. By [12, Proposition 4.2] $Sat_{\tau_1}(E)$ consist of the τ_1 -closed submodules of E . Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ker(\rho) & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \scriptstyle F\psi & & \\ 0 & \longrightarrow & Ker(\rho) & \longrightarrow & E & \longrightarrow & \tau_1 F & \longrightarrow & 0 \end{array}$$

Since τ_1 is perfect the localization functor is exact and $Ker(\rho)$ and E are τ_1 -closed, so the second row is exact. This implies F is τ_1 -closed. Thus ρ is an $\tau_1 R$ -morphism. Since $\tau_1 R$ is left hereditary and E is an injective $\tau_1 R$ -module then F is an injective $\tau_1 R$ -module. Thus F is an injective R -module.

By [6, Proposition 14A] every injective R -module is completely injective, thus R is left hereditary. □

Theorem 3.13. *Let R be a (simple) left QI-ring. Suppose that $Gdim(R) = n$ and let $\{\tau_i\}_{i=1}^n$ be the Gabriel filtration in R -tors. Then the following conditions are equivalent:*

1. *Every τ_i is perfect*
2. *R is left hereditary.*

Proof. \Rightarrow By induction over n .

If $n = 1$ then R is semisimple, and thus R is hereditary.

Suppose the result is true for all left QI-rings R with $Gdim(R) < n$ such that every element in the Gabriel filtration is perfect. Let R be a left QI-ring with $Gdim(R) = n$. By hypothesis $\tau_1 = \tau_{ss}$ is perfect.

By Proposition 3.10 $\tau_1 R$ is a left QI-ring and by Lemma 3.5 $Gdim(\tau_1 R) < n$. Since the Gabriel filtration in $\tau_1 R$ -tors is $\{\mathfrak{q}(\tau_i) | 1 \leq i < n\}$ then, by

Lemma 3.2 we can apply the induction hypothesis. Hence $\tau_1 R$ is left hereditary. Thus by Theorem 3.12 R is left hereditary.

\Leftarrow . If R is left hereditary and left QI-ring then every hereditary torsion theory is perfect [12, XI, Corollary 3.6]. \square

Definition 3.14. An R -module M satisfies the restricted left socle condition (RLS) if for any essential submodule $N \neq M$, the factor module M/N has non zero socle.

Proposition 3.15. Let R be a simple left QI-ring which is non semisimple. The following are equivalent:

1. R satisfies RLS.
2. $\tau_{ss} = \tau_g$
3. $Gdim(R) = 2$.
4. There exists a non singular indecomposable completely injective module E which is τ_{ss} -cocritical.

Proof. $1 \Rightarrow 2$. Since R is non semisimple, then by Remark 3.11 $\tau_{ss} \leq \tau_g$. Now let M be a singular module and $0 \neq m \in M$. Then $(0 : m)$ is an essential left ideal of R . Thus $0 \neq Soc(R/(0 : m)) = Soc(Rm)$. This implies that $Soc(M) \leq_e M$ but $Soc(M)$ is a direct summand, so $M = Soc(M)$.

$2 \Rightarrow 3$. If $\{\tau_i | i \geq 0\}$ is the Gabriel filtration in R -tors then $\tau_1 = \tau_{ss} = \tau_g$. Since R is simple τ_g is a coatom in R -tors, thus $\tau_2 = \chi$.

$3 \Rightarrow 4$. If $Gdim(R) = 2$, then the Gabriel filtration in R -tors is $\{\xi, \tau_{ss}, \chi\}$. Therefore

$$\chi = \tau_{ss} \vee \bigvee \{\xi(N) | N \text{ } \tau_{ss}\text{-cocritical}\}$$

Assume that every τ_{ss} -cocritical module is singular, then $\chi \leq \tau_g$, this is a contradiction. Hence, there exists a non singular τ_{ss} -cocritical module N . Since N is cocritical, it is uniform and so $E(N)$ is a non singular indecomposable injective module. By [8, proposition 2.1] $E(N)$ is also τ_{ss} -cocritical, thus we are done.

$4 \Rightarrow 1$. Since R is a simple ring then R is a prime ring. By [3, Corollary 2.15], E is up to isomorphism the only non singular indecomposable injective module. Now, by [3, Theorem 2.20] $E(R) \cong E^k$ for some $k > 0$; since E is completely injective then $E(R)$ so does by [6, Proposition 14A]. Let $I \leq_e R$, then $I \leq_e E(R)$ and so $E(R)/I$ is semisimple. Thus R/I is semisimple. \square

Remark 3.16. In [6, Theorem 17] it is constructed an indecomposable injective module E such that E is non semisimple and satisfies RLS . Then $Soc(E) = 0$, that is, E is τ_{ss} -torsion free. Since E is uniform and satisfies RLS then E is τ_{ss} -cocritical. Thus if E is nonsingular then E satisfies condition 4 of Proposition 3.15.

As Corollary we have the next result due to C. Faith [6, Theorem 18]

Corollary 3.17. *Any left QI-ring R with restricted left socle condition is left hereditary.*

Proof. By Proposition 3.15 R has $Gdim(R) = 2$. Hence the Gabriel filtration in R -tors is $\{\xi, \tau_g, \chi\}$ where ξ and χ are the least and greatest elements of R -tors respectively. The element $\tau_g \in R$ -tors is the Goldie's torsion theory and it is perfect because R is left noetherian [12, Proposition 2.12 and Proposition 3.4]. Thus, by Theorem 3.13 R is left hereditary. \square

Lemma 3.18. *Let $R = R_1 \times \cdots \times R_n$ be a finite product of rings. Then R satisfies RLS if and only if R_i satisfies RLS for all $1 \leq i \leq n$.*

Proof. \Rightarrow . Let $1 \leq i \leq n$ and I_i an essential left ideal of R_i . Then $I = R_1 \times \cdots \times I_i \times \cdots \times R_n$ is an essential left ideal of R . By hypothesis, R/I contains a simple R -module S , and we have that $R/I \cong R_i/I_i$. Thus S is a simple R_i -module and it can be embedded in R_i/I_i .

\Leftarrow . Let I be an essential left ideal of R , then $I = I_1 \times \cdots \times I_n$ with I_i an essential left ideal of R_i . By hypothesis R_i/I_i contains a simple R_i -module and we have that $R/I \cong (R_1/I_1) \oplus \cdots \oplus (R_n/I_n)$. Thus R/I contains a simple R -module. \square

Remark 3.19. Let $R = R_1 \times \cdots \times R_n$ be a product of rings. Notice that E is a non singular indecomposable injective R -module then E is a non singular indecomposable injective R_i -module for some $1 \leq i \leq n$. On the other hand, if E_i is a non singular indecomposable injective R_i -module then E_i is a non singular indecomposable injective R -module.

Theorem 3.20. *Let $R = R_1 \times \cdots \times R_n$ be a left QI-ring such that each R_i is a simple left QI-ring and non semisimple. The following conditions are equivalent:*

1. R satisfies RLS .

2. For each $1 \leq i \leq n$ there exists a non singular indecomposable injective R_i -module E_i which are τ_{ss} -cocritical as R -modules.
3. $Gdim(R) = 2$.
4. $\tau_{ss} = \tau_g$ in R -tors.

Proof. $1 \Rightarrow 2$. Since R satisfies RLS then by Lemma 3.18 each R_i satisfies RLS . Hence by Proposition 3.15 there exist a non singular indecomposable injective R_i -module E_i which satisfies RLS .

$2 \Rightarrow 3$. Let $\{\tau_j\}$ be the Gabriel filtration in R -tors. By Remark 3.19 each E_i is a non singular indecomposable injective R -module, so each E_i is τ_{ss} -torsion free. Since each E_i is τ_{ss} -cocritical then

$$\tau_{ss} \vee \bigvee \xi(E_i) \leq \tau_2$$

Now, if E is a non singular indecomposable injective R -module then E is a non singular indecomposable injective R_i -module for some $1 \leq i \leq n$. But since R_i is simple and hence a prime ring, by [3, Corollary 2.20] $E \cong E_i$. Thus all non singular indecomposable injective R -modules, up to isomorphism, are E_1, \dots, E_n . Again by [3, Corollary 2.20] $\hat{R} \cong E_1^{k_1} \oplus \dots \oplus E_n^{k_n}$ where \hat{R} denotes the injective hull of R for some natural numbers k_1, \dots, k_n . Thus $\tau_{ss} \vee \bigvee \xi(E_i) = \chi$. So, $Gdim(R) = 2$.

$3 \Rightarrow 4$. If $Gdim(R) = 2$ then the Gabriel filtration in R -tors is $\{\xi, \tau_{ss}, \chi\}$. Since every R_i is a simple left QI-ring non semisimple, then all simple R -modules are singular. Thus $\tau_{ss} \leq \tau_g$.

Suppose $\tau_{ss} < \tau_g$ then there exists C such that C is τ_{ss} -cocritical and τ_g -torsion. If $c \in C$, $ann(c) \leq_e R$, and $ann(c) = I_1 \times \dots \times I_n$ with $I_i \leq_e R_i$. Hence R/I_i is singular and we have a monomorphism

$$R/I \cong (R_1/I_1) \oplus \dots \oplus (R_n/I_n) \rightarrow C$$

Since every R_i is a simple left QI-ring and R_i/I_i is singular then by Proposition 3.15 R_i/I_i is a semisimple R_i module, hence it is semisimple as R -module. Thus C is semisimple and $\tau_{ss} = \tau_g$.

$4 \Rightarrow 1$. Let $I \leq_e R$, then R/I is τ_g -torsion then it is τ_{ss} -torsion. This implies that R/I contains a simple R -module. Thus R satisfies RLS . \square

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